Ex 1. 直接利用 Hint 即可

注意,题目有两问!!

Ex 2.

I. Stein 1.2: Cantor set described in ternary expansions

Some notations are shown as follows:

On construction of Cantor set $\mathcal{C}$: Let $C_0 = [0,1]$. This interval is divided into three parts: the middle third open interval of C_0 is $E_{1,1} = (1/3, 2/3)$ which is excluded, while $I_{1,1} = [0, 1/3]$, $I_{1,2} = [2/3, 1]$ is included to obtain $C_1 = I_{1,1} \cup I_{1,2}$. Generally, when we get $C_k = \cup_{i=1}^{2^{k-1}} I_{k,i}$, each $I_{k,i}$ is divided into three parts with the middle open one denoted as $E_{k+1,i}$, the other two are $I_{k+1,2i-1}, I_{k+1,2i}$. Then $\mathcal{C} := \cap_{k=0}^{\infty} C_k$.

A. *Cantor sets are points represented in 0 and 2.*

Proof: Notice that $x \notin E_{1,1}$ (*i.e.*, $x \in C_1$) if and only if x has a decomposition with $a_1 \neq 1$. By deduction, we claim that C_k consists of x that has a decomposition with $a_j \neq 1, j \leq k$. The definition of $\mathcal{C}$ completes the proof. ∎

B. *Well-definedness and continuity of Cantor-Lebesgue function.*

Proof: First, to show the well-definedness, it is sufficient to show that the ternary expansion of 0 and 2 is unique. Suppose that

$$x = \sum_{k=1}^{\infty} a_k 3^{-k} = \sum_{k=1}^{\infty} b_k 3^{-k}, \quad a_k, b_k \in \{0, 2\}.$$

Let $k_0 := \inf\{k : a_k \neq b_k\} < \infty$, and WLOG (without loss of generality), $a_{k_0} = 0, b_{k_0} = 2$. Then

$$0 = \sum_{k=1}^{\infty} (b_k - a_k) 3^{-k} \geq 2 \cdot 3^{-k_0} - \sum_{k=k_0+1}^{\infty} 2 \cdot 3^{-k} = 3^{-k_0},$$

which is a contradiction! This argument also shows that $\forall |x - y| < 3^{-k_0}, x, y \in \mathcal{C}$, the first k_0 expansions are the same.

Second, the continuity. $\forall \epsilon > 0$, $\exists k_0$, *s.t.* $2^{-k_0} < \epsilon$. Therefore, for any $x \in \mathcal{C}$, choose $\delta = 3^{-k_0}$, then $\forall y \in \mathcal{C}, |x-y| < \delta$, the first k_0 expansions of x and y are the same, thus $|F(x) - F(y)| \leq \sum_{k_0+1}^{\infty} 2^{-k} = 2^{-k_0} < \epsilon$. From the definition, we know that F is continuous.

Moreover, $F(0) = 0, F(1) = 1$ follows directly from $0 = \sum_{k=1}^{\infty} 0 \cdot 3^{-k}$ and $1 = \sum_{k=1}^{\infty} 2 \cdot 3^{-k}$. ∎

C. *Surjectiveness of Cantor-Lebesgue function.*

Proof: Every $y \in [0,1]$ has a binary expression $y = \sum_{k=1}^{\infty} b_k \cdot 2^{-k}, b_k \in \{0, 1\}$, from which we can recover $x = \sum_{k=1}^{\infty} 2b_k \cdot 3^{-k} \in \mathcal{C}$. ∎

D. *Continuity of extended Cantor-Lebesgue function.*

Proof: Note that $F|_{\mathcal{C}}$ is non-decreasing and the extended definition can be interpreted as $F(x) := F(\sup\{y : y \in \mathcal{C}, y \leq x\} = \sup\{F(y) : y \in \mathcal{C}, y \leq x\}$ for any $x \in [0,1]$, or $F(x) := \inf\{F(y) : y \in \mathcal{C}, y \geq x\}$ due to the monotonicity of $F|_{\mathcal{C}}$ and closedness of $\mathcal{C}$.

$\forall \epsilon > 0, x \in [0,1]$, choose $\delta = 3^{-k_0}$ same as in (I-B), (*s.t.* $2^{-k_0} < \epsilon$,) then $\forall |x - y| < \delta$, if $x, y \in \mathcal{C}$, $|F(x) - F(y)| \leq \sum_{k_0+1}^{\infty} 2^{-k} = 2^{-k_0} < \epsilon$; else, WLOG $y > x$, let $x' = \inf\{z : z \in \mathcal{C}, z \geq x\}$ and $y' = \inf\{z : z \in \mathcal{C}, z \leq y\}$, then $|F(x) - F(y)| = |F(x') - F(y')| < \epsilon$. From the definition, F is continuous. ∎

Ex 4. (d) 利用 Ex2 中的 Cantor-Lebesgue 函数.

得到 $\mathcal{C} \to [0,1]$ 满射.

$\updownarrow$ 1-1

C.

本题也可直接 $m(\mathcal{C}) > 0$.

Ex 14

III. Stein 1.14: Outer Jordan content - finite covering intervals

A. $J_*(E) = J_*(\bar{E})$.

Proof: If $\bar{E} \subset \cup_{j=1}^{N} I_j$, then $E \subset \bar{E} \subset \cup_{j=1}^{N} I_j$, thus $J_*(E) \leq J_*(\bar{E})$. If $E \subset \cup_{j=1}^{N} I_j$, then $\bar{E} \subset \cup_{j=1}^{N} \bar{I}_j$ with $\sum |I_j| = \sum |\bar{I}_j|$, thus $J_*(E) \geq J_*(\bar{E})$. ∎

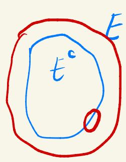
B. $J_*(E) = 1, m_*(E) = 0$.

$E = \mathbb{Q} \cap [0,1]$ is an example.

Ex 25

IX. Stein 1.25: Equivalent definition of measurability by inner closed approximation

Proof: In this problem, we call our original definition as *open-measurable* and the alternative as *close-measurable*. If E is close-measurable, then E^c is open-measurable, since $E - F = F^c - E^c$. From Property 5 (P18), E is open-measurable. On the other hand, if E is open-measurable, from Theorem 3.4 (ii) (P21), E is close-measurable. ∎



$F = O^c$

$E - F = O - E^c$.

Lemma 1.2 *If $R, R_1, \ldots, R_N$ are rectangles, and $R \subset \bigcup_{k=1}^{N} R_k$, then*

$$|R| \leq \sum_{k=1}^{N} |R_k|.$$

参考 Stein 英文版 P5-6

HW: Lebesgue 可测 $\Leftrightarrow$ Carathéodory 可测.

Pf: 记 Carathéodory 可测集构成的集合为 $\mathcal{M}_1$, Lebesgue 可测集构成的集合为 $\mathcal{M}_2$.

①. 若 $E \in \mathcal{M}_1$. 由 m_* 满足 $m_*(E) = \inf\{m_*(O) : O \text{ is open}, O \supset E\}$ (P13 Observation 3),

对 $\forall \frac{1}{n}$, $\exists$ open set $O_n \supset E$ s.t. $m_*(O_n) < m_*(E) + \frac{1}{n}$.

令 $G = \bigcap_{n=1}^{\infty} O_n$. G 为 G_δ 集. 则 $E \subset G$ 且 $m_*(E) = m_*(G)$. ("$\leq$"✓ "$\geq$" $m_*(G) < m_*(E) + \frac{1}{n}, \forall n \Rightarrow$ ✓)

由 $E \in \mathcal{M}_1 \Rightarrow m_*(G) = m_*(G \cap E) + m_*(G \cap E^c) = m_*(E) + m_*(G \setminus E) \Rightarrow m_*(G \setminus E) = 0$

$\Rightarrow G \setminus E \in \mathcal{M}_2$. 又 $G \in \mathcal{M}_2$. 故 $E = G \setminus (G \setminus E) \in \mathcal{M}_2$. $\Rightarrow \mathcal{M}_1 \subset \mathcal{M}_2$.

②. 若 $E \in \mathcal{M}_2$. 则 $\forall A \subset \mathbb{R}^d$, 取 G_δ 集 G s.t. $A \subset G$ 且 $m_*(A) = m(G)$. (取法同①. G 称为 A 的等测包)

则 $m_*(A \cap E) + m_*(A \cap E^c) \leq m_*(G \cap E) + m_*(G \cap E^c) = m(G \cap E) + m(G \cap E^c) = m(G) = m_*(A)$.

故 $E \in \mathcal{M}_1 \Rightarrow \mathcal{M}_2 \subset \mathcal{M}_1$.

(注. 以上均假设 $m_*(E) < \infty$. 否则记 $E = \bigcup_{n=1}^{\infty} (E \cap B(0,n))$ 即可). □